

Reconstruction of the core convex topology and its applications in vector optimization and convex analysis

Ashkan Mohammadi

Wayne State University, Detroit, MI 48202;
ashkan.mohammadi@wayne.edu

&

Majid Soleimani-damaneh

University of Tehran, Tehran, Iran;
soleimani@khayam.ut.ac.ir

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Abstract

In this paper, the core convex topology on a real vector space X , which is constructed just by X operators, is investigated. This topology, denoted by τ_c , is the strongest topology which makes X into a locally convex space. It is shown that some algebraic notions (*closure and interior*) existing in the literature come from this topology. In fact, it is proved that algebraic interior and vectorial closure notions, considered in the literature as replacements of topological interior and topological closure, respectively, in vector spaces not necessarily equipped with a topology, are actually nothing else than the interior and closure with the respect to the core convex topology. We reconstruct the core convex topology using an appropriate topological basis which enables us to characterize its open sets.

Furthermore, it is proved that (X, τ_c) is not metrizable when X is infinite-dimensional, and also it enjoys the Hine-Borel property. Using these properties, τ_c -compact sets are characterized and a characterization of finite-dimensionality is provided. Finally, it is shown that the properties of the core convex topology lead to directly extending various important results in convex analysis and vector optimization from topological vector spaces to real vector spaces.

Keywords: *Core convex topology, Functional Analysis, Vector optimization, Convex Analysis.*

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1 Introduction

Convex Analysis and Vector Optimization under real vector spaces, without any topology, have been studied by various scholars in recent years [1, 2, 3, 8, 16, 18, 19, 20, 29, 30]. Studying these problems opens new connections between Optimization, Functional Analysis, and Convex analysis. Since (relative) interior and closure notions play important roles in many convex analysis and optimization problems [4, 18, 21], due to the absence of topology, we have to use some algebraic concepts.

To this end, the concepts of *algebraic (relative) interior* and *vectorial closure* have been investigated in the literature, and many results have been provided invoking these algebraic concepts; see e.g. [1, 2, 3, 11, 16, 18, 20, 23, 24, 29, 30] and the references therein. The main aim of this paper is to unifying vector optimization in real vector spaces with vector optimization in topological vector spaces.

In this paper, core convex topology (see [10, 19]) on an arbitrary real vector space, X , is dealt with. Core convex topology, denoted by τ_c , is the strongest topology which makes a real vector space into a locally convex space (see [10, 19]). The topological dual of X under τ_c coincides with its algebraic dual [10, 19]. It is quite well known that when a locally convex space is given by a family of seminorms, the locally convex topology is deduced in a standard way and vice versa. In this paper, τ_c is reconstructed by a topological basis. It is known that algebraic (relative) interior of a convex set is a topological notion which can be derived from core convex topology [10, 19]. We provide a formula for τ_c -interior of an arbitrary (nonconvex) set with respect to the algebraic interior of its convex components. Furthermore, we show that vectorial closure is also a topological notion coming from core convex topology (under mild assumptions). According to these facts, various important results, in convex analysis and vector optimization can be extended easily from topological vector spaces (TVSs) to real vector spaces. Some such results are addressed in this paper. After providing some basic results about open sets in τ_c , it is proved that, X is not metrizable under τ_c topology if it is infinite-dimensional. Also, it is shown that (X, τ_c) enjoys the Hine-Borel property. A characterization of open sets in terms of there convex components is given. Moreover, τ_c -convergence as well as τ_c -compactness are characterized.

The rest of the paper unfolds as follows. Section 2 contains some preliminaries and Section 3 is devoted to the core convex topology. Section 4 concludes the paper by addressing some results existing in vector optimization and convex analysis literature which can be extended from TVSs to real vector spaces, utilizing the results given in the present paper.

2 Preliminaries

Throughout this paper, X is a real vector space, A is a subset of X , and $K \subseteq X$ is a nontrivial nonempty ordering convex cone. K is called pointed if $K \cap (-K) = \{0\}$. $\text{cone}(A)$, $\text{conv}(A)$, and $\text{aff}(A)$ denote the cone generated by A , the convex hull of A , and the affine hull of A , respectively.

For two sets $A, B \subseteq X$ and a vector $\bar{a} \in X$, we use the following notations:

$$\begin{aligned} A \pm B &:= \{a \pm b : a \in A, b \in B\}, \\ \bar{a} \pm A &:= \{\bar{a} \pm a : a \in A\}, \\ A \setminus B &:= \{a \in A : a \notin B\}. \end{aligned}$$

$P(X)$ is the set of all subsets of X and for $\Gamma \subseteq P(X)$,

$$\cup \Gamma := \{x \in X : \exists A \in \Gamma; x \in A\}$$

The algebraic interior of $A \subseteq X$, denoted by $\text{cor}(A)$, and the relative algebraic interior of A , denoted by $\text{icr}(A)$, are defined as follows [16, 19]:

$$\begin{aligned} \text{cor}(A) &:= \{x \in A : \forall x' \in X, \exists \lambda' > 0; \forall \lambda \in [0, \lambda'], x + \lambda x' \in A\}, \\ \text{icr}(A) &:= \{x \in A : \forall x' \in L(A), \exists \lambda' > 0; \forall \lambda \in [0, \lambda'], x + \lambda x' \in A\}, \end{aligned}$$

where $L(A) = \text{span}(A - A)$ is the linear hull of $A - A$. When $\text{cor}(A) \neq \emptyset$ we say that A is solid; and we say that A is relatively solid if $\text{icr}(A) \neq \emptyset$. The set A is called algebraic open if $\text{cor}(A) = A$. The set of all elements of X which do not belong to $\text{cor}(A)$ and $\text{cor}(X \setminus A)$ is called the algebraic boundary of A . The set A is called algebraically bounded, if for every $x \in A$ and every $y \in X$ there is a $\lambda > 0$ such that

$$x + ty \notin A \quad \forall t \in [\lambda, \infty).$$

If A is convex, then there is a simple characterization of $\text{icr}(A)$ as follows: $a \in \text{icr}(A)$ if and only if for each $x \in A$ there exists $\delta > 0$ such that $a + \lambda(a - x) \in A$, for all $\lambda \in [0, \delta]$.

Lemma 2.1. *Let $\{e_i\}_{i \in I}$ be a vector basis for X , and $A \subseteq X$ be nonempty and convex. $a \in \text{cor}(A)$ if and only if for each $i \in I$ there exists scalar $\delta_i > 0$ such that $a \pm \delta_i e_i \in A$.*

Proof. Assume that for each $i \in I$ there exists scalar $\delta_i > 0$ such that $a \pm \delta_i e_i \in A$. Let $d \in X$. There exist a finite set $J \subseteq I$ and positive scalars $\lambda_j, \mu_j, \delta_j$, $j \in J$ such that

$$d = \sum_{j \in J} \lambda_j e_j - \sum_{j \in J} \mu_j e_j, \quad a \pm \delta_j e_j \in A, \quad \forall j \in J.$$

Let $m := \text{Card}(J)$ and $\delta > 0$. Considering $\delta \in (0, \min\{\frac{1}{2m}, \frac{\delta_j}{2m\lambda_j}\})$, we have

$$a + 2m\delta\lambda_j e_j \in [a, a + \delta_j e_j], \quad a - 2m\delta\mu_j e_j \in [a - \mu_j e_j, a], \quad \forall j \in J,$$

where $[x, y]$ stands for the line segment joining x, y . Since A is convex,

$$a + 2m\delta\lambda_j e_j \in A, \quad a - 2m\delta\mu_j e_j \in A, \quad \forall j \in J,$$

and then, due to the convexity of A again,

$$\frac{a}{2} + \sum_{j \in J} \delta\lambda_j e_j \in \frac{A}{2}, \quad \frac{a}{2} - \sum_{j \in J} \delta\mu_j e_j \in \frac{A}{2}.$$

This implies

$$a + \sum_{j \in J} \delta\lambda_j e_j - \sum_{j \in J} \delta\mu_j e_j \in A,$$

which means $a + \delta d \in A$. Furthermore, the convexity of A guarantees that $a + \lambda d \in A$ for all $\lambda \in [0, \delta]$. Thus $a \in \text{cor}(A)$. The converse is obvious. \square

Some basic properties of the algebraic interior are summarized in the following lemmas. The proof of these lemmas can be found in the literature; see e.g. [1, 2, 10, 16, 19].

Lemma 2.2. *Let A be a nonempty set in real vector space X . Then the following propositions hold true:*

1. *If A is convex, then $\text{cor}(\text{cor}(A)) = \text{cor}(A)$,*
2. *$\text{cor}(A \cap B) = \text{cor}(A) \cap \text{cor}(B)$,*
3. *$\text{cor}(x + A) = x + \text{cor}(A)$ for each $x \in X$,*
4. *$\text{cor}(\alpha A) = \alpha \text{cor}(A)$, for each $\alpha \in \mathbb{R} \setminus \{0\}$,*
5. *If $0 \in \text{cor}(A)$, then A is absorbing (i.e. $\text{cone}(A) = X$).*

Lemma 2.3. *Let $K \subseteq X$ be a convex cone. Then the following propositions hold true:*

- i. *If $\text{cor}(K) \neq \emptyset$, then $\text{cor}(K) \cup \{0\}$ is a convex cone,*
- ii. *$\text{cor}(K) + K = \text{cor}(K)$,*
- iii. *If $K, C \subseteq X$ are convex and relatively solid, then $\text{icr}(K) + \text{icr}(C) = \text{icr}(K + C)$.*
- iv. *If $f : X \rightarrow \mathbb{R}$ is a convex (concave) function, then f is τ_c -continuous.*

Although the (relative) algebraic interior is usually defined in vector spaces without topology, in some cases it might be useful under TVSs too. It is because the algebraic (relative) interior can be nonempty while (relative) interior is empty. The algebraic (relative) interior preserves most of the properties of (relative) interior.

Let Y be a real topological vector space (TVS) with topology τ . We denote this space by (Y, τ) . The interior of $A \subseteq Y$ with respect to topology τ is denoted by $\text{int}_\tau(A)$. A vector $a \in A$ is called a relative interior point of A if there exists some open set U such that $U \cap \text{aff}(A) \subseteq A$. The set of relative interior points of A is denoted by $\text{ri}_\tau(A)$.

Lemma 2.4. *Let (Y, τ) be a real topological vector space (TVS) and $A \subseteq Y$. Then $\text{int}_\tau(A) \subseteq \text{cor}(A)$. If furthermore A is convex and $\text{int}_\tau(A) \neq \emptyset$, then $\text{int}_\tau(A) = \text{cor}(A)$.*

The algebraic dual of X is denoted by X' , and $\langle \cdot, \cdot \rangle$ exhibits the duality pairing, i.e., for $l \in X'$ and $x \in X$ we have $\langle l, x \rangle := l(x)$. The nonnegative dual and the positive dual of K are, respectively, defined by

$$K^+ := \{l \in X' : \langle l, a \rangle \geq 0, \quad \forall a \in K\},$$

$$K^{+s} := \{l \in X' : \langle l, a \rangle > 0, \quad \forall a \in K \setminus \{0\}\}.$$

If K is a convex cone with nonempty algebraic interior, then $\text{cor}(K) = \{x \in K : \langle l, x \rangle > 0, \quad \forall l \in K^+ \setminus \{0\}\}$.

The vectorial closure of A , which is considered instead of closure in the absence of topology, is defined by [1]

$$\text{vcl}(A) := \{b \in X : \exists x \in X ; \forall \lambda' > 0, \exists \lambda \in [0, \lambda'] ; b + \lambda x \in A\}.$$

A is called vectorially closed if $A = \text{vcl}(A)$.

3 Main results

This section is devoted to constructing core convex topology via a topological basis. Formerly, the core convex topology was constructed via a family of separating semi-norms on X ; see [19]. In this section, we are going to construct core convex topology directly by characterizing its open sets. The first step in constructing a topology is defining its basis. The following definition and two next lemmas concern this matter.

Definition 3.1. [22] Let F be a subset of $P(X)$, where $P(X)$ stands for the power set of X . Then, F is called a topological basis on X if $X, \emptyset \in F$ and moreover, the intersection of each two members of F can be represented as union of some members of F .

The following lemma shows how a topology is constructed from a topological basis.

Lemma 3.1. *If F is a topological basis on X , then the collection of all possible unions of members of F is a topology on X .*

Lemma 3.2 provides the basis of the topology which we are looking for. The proof of this lemma is clear according to Lemma 2.2.

Lemma 3.2. *The collection*

$$\mathfrak{B} := \{A \subseteq X : \text{cor}(A) = A, \text{conv}(A) = A\}$$

is a topological basis on X .

Now, we denote the topology generated by

$$\mathfrak{B} := \{A \subseteq X : \text{cor}(A) = A, \text{conv}(A) = A\}$$

by τ_c ; more precisely

$$\tau_c := \{\cup \Gamma \in P(X) : \Gamma \subseteq \mathfrak{B}\}.$$

The following theorem shows that τ_c is the strongest topology which makes X into a locally convex TVS. This theorem has been proved in [19] using a family of semi-norms defined on X . Here, we provide a different proof.

Theorem 3.1.

- i. (X, τ_c) is a locally convex TVS;
- ii. τ_c is the strongest topology which makes X into a locally convex space.

Proof. By Lemmas 3.1 and 3.2, τ_c is a topology on X .

Proof of part i: To prove this part, we should show that (X, τ_c) is a Hausdorff space, and two operators addition $+: X \times X \rightarrow X$ and scalar multiplication $+: \mathbb{R} \times X \rightarrow X$ are τ_c -continuous.

Continuity of addition: Let $x, y \in X$ and let V be a τ_c -open set containing $x + y$. We should find two τ_c -open sets V_x and V_y containing x and y , respectively, such that $V_x + V_y \subseteq V$. Since \mathfrak{B} is a basis for τ_c , there exists $A \in \mathfrak{B}$ such that

$$x + y \in A \subseteq V.$$

Defining

$$V_x := \frac{1}{2}(A - x - y) + x \quad \text{and} \quad V_y := \frac{1}{2}(A - x - y) + y,$$

by Lemma 2.2, we conclude that $V_x, V_y \in \mathfrak{B}$ and $V_x + V_y = A \subseteq V$. Convexity of A , implies that V_x and V_y are the desired τ_c -open sets, and hence the addition operator is τ_c -continuous.

Continuity of scalar multiplication: Let $x \in X$, $\alpha \in \mathbb{R}$, and V be a τ_c -open set containing αx . without lose of generality, assume that $V \in \mathfrak{B}$. We must show that there exist $\varepsilon > 0$ and a τ_c -open set V_x containing x such that

$$(\alpha - \varepsilon, \alpha + \varepsilon)V_x \subseteq V.$$

Since $\alpha x \in V = \text{cor}(V)$, by considering $d := \pm x$ in the definition of algebraic interior, there exists $\delta > 0$ such that

$$\alpha x + \lambda x \in V, \quad \lambda \in (-\delta, \delta).$$

Define

$$U := (V - \alpha x) \cap -(V - \alpha x).$$

We get $U = -U$, $0 \in U$, and by Lemma 2.2, $U \in \mathfrak{B}$. Furthermore, U is balanced (i.e. $\alpha U \subseteq U$ for each $\alpha \in [-1, 1]$), because U is convex and $0 \in U$. Now, we claim that

$$(\alpha - \frac{\delta}{2}, \alpha + \frac{\delta}{2}) \left(\frac{1}{2|\alpha| + \delta} U + x \right) \subseteq V. \quad (1)$$

To prove (1), let $\alpha + t \in (\alpha - \frac{\delta}{2}, \alpha + \frac{\delta}{2})$ with $|t| < \frac{\delta}{2}$. Therefore

$$\left| \frac{\alpha + t}{2|\alpha| + \delta} \right| = \frac{1}{2} \left| \frac{\alpha + t}{|\alpha| + \frac{\delta}{2}} \right| \leq \frac{1}{2} \frac{|\alpha| + |t|}{|\alpha| + \frac{\delta}{2}} \leq \frac{1}{2}$$

Thus,

$$\frac{\alpha + t}{2|\alpha| + \delta} U \subseteq \frac{1}{2} U.$$

Hence,

$$\begin{aligned} (\alpha + t) \left(\frac{1}{2|\alpha| + \delta} U + x \right) &= \frac{\alpha + t}{2|\alpha| + \delta} U + \alpha x + tx \subseteq \frac{1}{2} U + \alpha x + tx \\ &\subseteq \frac{1}{2} (V - \alpha x) + \alpha x + tx = \frac{1}{2} V + \frac{1}{2} (\alpha x + 2tx) \subseteq \frac{1}{2} V + \frac{1}{2} V = V. \end{aligned}$$

This proves (1). Setting $\varepsilon := \frac{\delta}{2}$ and $V_x := \frac{1}{2|\alpha| + \delta} U + x$ proves the continuity of the scalar multiplication operator.

Now, we show that (X, τ_c) is a Hausdorff space. To this end, suppose $x_0 \in X \setminus \{0\}$. Consider $f \in X'$ such that $f(x_0) = 2$, and set $A := \{x \in X : f(x) < 1\}$ and $B := \{x : f(x) > 1\}$. It is not difficult to see that $A, B \in \mathfrak{B}$ and $x_0 \in B$ while $0 \in A$. This implies that (X, τ_c) is a Hausdorff space.

ii. Let τ be an arbitrary topology on X which makes X into a locally convex space. Let B_τ be a locally convex basis of topology τ . For each $U \in B_\tau$ we have $\text{cor}(U) = \text{int}_\tau(U) = U$ (by Lemma 2.4) and hence $U \in \mathfrak{B}$. Thus we have $B_\tau \subseteq \mathfrak{B}$, which leads to $\tau \subseteq \tau_c$ and completes the proof. \square

The interior of $A \subseteq X$ with respect to τ_c topology is denoted by $\text{int}_c(A)$. The following theorem shows that the algebraic interior (i.e. cor) for convex sets is a topological interior coming from τ_c .

Theorem 3.2. [19, Proposition 6.3.1] *Let $A \subseteq X$ be a convex set. Then*

$$\text{int}_c(A) = \text{cor}(A).$$

Proof. Since (X, τ_c) is a TVS, $\text{int}_c A \subseteq \text{cor}(A)$; see Lemma 2.4. Since A is convex, $\text{cor}(A)$ is also convex, and furthermore $\text{cor}(\text{cor}(A)) = \text{cor}(A)$ (by Lemma 2.2). Hence, $\text{cor}(A) \in \mathfrak{B}$. Therefore, $\text{cor}(A) \subseteq \text{int}_c(A)$, because $\text{int}_c(A)$ is the biggest subset of A belonging to \mathfrak{B} . Thus $\text{int}_c(A) = \text{cor}(A)$, and the proof is completed. \square

Notice that the convexity assumption in Theorem 3.2 is essential; see Example 3.1.

The proof of the following result is similar to that of Theorem 3.2.

Theorem 3.3. *If A is a convex subset of X , then $\text{icr}(A) = \text{ri}_c(A)$, where $\text{ri}_c(A)$ denotes the relative interior of A with respect to the topology τ_c .*

It is seen that the convexity assumption plays a vital role in Theorems 3.2 and 3.3. In the following two results, we are going to characterize the τ_c -interior of an arbitrary (nonconvex) nonempty set with respect to the core of its convex components. Since $\text{int}_c(A) \in \tau_c$ and \mathfrak{B} is a basis for τ_c , the set $\text{int}_c(A)$ could be written as union of some subsets of A which are algebraic open.

Lemma 3.3. *Let A be a nonempty subset of real vector space X . Then A could be uniquely decomposed to the maximal convex subsets of A , i.e. $A := \bigcup_{i \in I} A_i$ where A_i , $i \in I$ are non-identical maximal convex subsets (not necessary disjoint) of A (Here, A_i sets are called convex components of A).*

Proof. For every $a \in A$, let $\Upsilon_a \subseteq P(A)$ be the set of all maximal convex subsets of A containing a (the nonemptiness of such Υ_a is derived from Zorn lemma). Set the index set $I := \bigcup_{a \in A} \Upsilon_a$ (this type of defining I enables us to avoid repetition). For every $i \in I$, define $A_i = i$. Hence, $A = \bigcup_{i \in I} A_i$ where A_i , $i \in I$ are non-identical maximal convex subsets of A . To prove the uniqueness of A_i 's, suppose $A = \bigcup_{j \in J} B_j$ such that B_j ($j \in J$) are non-identical maximal convex subsets of A . Let $j \in J$ and $x \in B_j$; then $B_j \in \Upsilon_x$, and hence there exists $i \in I$ such that $B_j = i = A_i$. This means $\{B_j : j \in J\} = \{A_i : i \in I\}$, and the proof is completed. \square

Theorem 3.4. *Let A be a nonempty subset of real vector space X . Then*

$$\text{int}_c(A) = \bigcup_{i \in I} \text{cor}(A_i),$$

where A_i , $i \in I$ are convex components of A

Proof. Let $a \in \text{int}_c(A)$. There exists $V \in \mathfrak{B}$ such that $a \in V \subseteq A$. Set $\Pi := \{B \subseteq A : V \subseteq B, B = \text{conv}(B)\}$. Clearly $V \in \Pi$ and hence $\Pi \neq \emptyset$. Furthermore it is easy to verify that each chain (totally ordered subset) in Π has an upper bound within Π . Therefore, using Zorn lemma, Π has a maximal element. Let B_* be maximal element of Π . Obviously, B_* is a convex component of A . It leads to the existence of an $i \in I$ such that $a \in V \subseteq B_* = A_i$. Therefore, $a \in \text{cor}(A_i)$. To prove the other side, suppose $a \in \text{cor}(A_i)$ for some $i \in I$. Since A_i is convex, by Theorem 3.2, $a \in \text{cor}(A_i) = \text{int}_c(A_i) \subseteq \text{int}_c(A)$. \square

Example 3.1. *Consider*

$$A := \{(x, y) \in \mathbb{R}^2 : y \geq x^2\} \cup \{(x, y) \in \mathbb{R}^2 : y \leq -x^2\} \cup \{(x, y) \in \mathbb{R}^2 : y = 0\}$$

as a subset of \mathbb{R}^2 . It can be seen that $(0, 0) \in \text{cor}(A)$, while $(0, 0) \notin \text{int}_c(A) = \text{int}_{\|\cdot\|_2}(A)$. However, $\text{int}_c(A) = \bigcup_{i=1}^4 \text{cor}(A_i) = \bigcup_{i=1}^4 \text{int}_{\|\cdot\|_2}(A_i)$, where A_i , $i = 1, 2, 3, 4$ are convex components of A as follows

$$\begin{aligned} A_1 &= \{(x, y) \in \mathbb{R}^2 : y \geq x^2\}, \quad A_2 = \{(x, y) \in \mathbb{R}^2 : y \leq -x^2\}, \\ A_3 &= \{(x, y) \in \mathbb{R}^2 : y = 0\}, \quad A_4 = \{(x, y) \in \mathbb{R}^2 : x = 0\}. \end{aligned}$$

\square

The following theorem shows that the topological dual of (X, τ_c) is the algebraic dual of X . In the proof of this theorem, we use the topology which X' induces on X . This topology, denoted by τ_0 , is as follows:

$$\tau_0 = \{\cup \Gamma \in P(X) : \Gamma \subseteq \Psi\},$$

where

$$\begin{aligned} \Psi &= \{A \subseteq X : A = f_1^{-1}(I_1) \cap f_2^{-1}(I_2) \cap \dots \cap f_n^{-1}(I_n) \text{ for some } n \in \mathbb{N}, \text{ some} \\ &\quad \text{open intervals } I_1, I_2, \dots, I_n \subseteq \mathbb{R} \text{ and some } f_1, f_2, \dots, f_n \in X'\}. \end{aligned}$$

Theorem 3.5. $[10] (X, \tau_c)^* = X'$.

Proof. Let τ_0 denote the topology which X' induces on X . By [26, Theorem 3.10], τ_0 makes X into a locally convex TVS and $(X, \tau_0)^* = X'$. By Theorem 3.1, $\tau_0 \subseteq \tau_c$. Hence $(X, \tau_c)^* = X'$. \square

The following result provides a characterization of finite-dimensional spaces utilizing τ_c and the topology induced by X' on X .

Theorem 3.6. X is finite-dimensional if and only if $\tau_0 = \tau_c$.

Proof. Assume that X is finite-dimensional. Since there is only one topology on X which makes this space a TVS, we have $\tau_0 = \tau_c$.

To prove the converse, by indirect proof assume that $\dim(X) = \infty$. Let $\beta = \{x_i\}_{i \in I}$, be an ordered basis of X ; and $[x]_\beta$ denote the vector of coordinates of $x \in X$ with respect to the basis β . It is easy to show that, $[x]_\beta \in l_1(I)$ for each $x \in X$, where

$$l_1(I) = \{\{t_i\}_{i \in I} \subseteq \mathbb{R} : \|\{t_i\}\|_1 = \sum_{i \in I} |t_i| < \infty\}.$$

Define $\Omega : X \rightarrow \mathbb{R}$ by $\Omega(x) = \|[x]_\beta\|_1$. Since $\|\cdot\|_1$ is a norm on $l_1(I)$, thus Ω is also a norm on X . We denote this norm by $\|\cdot\|_0$.

Since τ_c is the strongest locally convex topology on X , we have $B_1^{\|\cdot\|_0}(0) \in \tau_c$, where $B_1^{\|\cdot\|_0}(0)$ stands for the unit ball with $\|\cdot\|_0$. On the other hand, every τ_0 -open set containing origin, contains an infinite-dimensional subspace of X (see [26]). Hence, $B_1^{\|\cdot\|_0}(0) \notin \tau_0$. This implies $\tau_0 \neq \tau_c$, and the proof is completed. \square

Theorem 3.7 demonstrates that the vectorial closure (vcl) for relatively solid convex sets, in vector space X , is a topological closure coming from τ_c . The closure of $A \subseteq X$ with respect to τ_c is denoted by $cl_c(A)$.

Theorem 3.7. *Let $A \subseteq X$ be a convex and relatively solid set. Then $vcl(A) = cl_c(A)$.*

Proof. Since (X, τ_c) is a TVS, it is easy to verify that $vcl(A) \subseteq cl_c(A)$. To prove the converse, let $x \in cl_c(A)$ and $a \in icr(A)$. Without loss of generality, we assume $a = 0$, and then we have $Y := span(A) = aff(A) = L(A)$, and $0 \in cor^Y(A)$, where $cor^Y(A)$ stands for the algebraic interior of A with respect to the subspace Y . We claim that $x \in Y$. To show this, assume that $x \notin Y$; then there exists a functional $f \in X'$ such that $f(x) = 1$ and $f(y) = 0$ for each $y \in Y$. Therefore the set

$$U := \{z \in X : f(z) > \frac{1}{2}\}$$

is a τ_c -open neighborhood of x with $U \cap A = \emptyset$, which contradicts $x \in cl_c(A)$. Now, we restrict our attention to subspace Y . By Theorem 3.2, $a \in int_c^Y(A)$, and thus, by [18, Lemma 1.32], $(x, a) \subseteq int_c^Y(A) \subseteq A$, which means $x \in vcl(A)$. Therefore $cl_c(A) \subseteq vcl(A)$, and the proof is completed. \square

Corollary 3.1. *Let A be a nonempty subset of X with finite many convex components. Then*

$$cl_c(A) = \bigcup_{i=1}^n vcl(A_i)$$

where A_i , $i = 1, 2, \dots, n$ are convex components of A .

Proof. According to Theorem 3.7, we have

$$cl_c(A) = \bigcup_{i=1}^n cl_c(A_i) = \bigcup_{i=1}^n vcl(A_i).$$

\square

The equality $cl_c(A) = \bigcup_{i \in I} vcl(A_i)$ may not be true in general; even if each convex component, A_i , is relatively solid. For example, consider \mathbb{Q} as the set of rational numbers in \mathbb{R} . The convex components of \mathbb{Q} are singleton, which are relatively solid. Therefore $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$, while

$$\bigcup_{q \in \mathbb{Q}} vcl(\{q\}) = \bigcup_{q \in \mathbb{Q}} \{q\} = \mathbb{Q} \neq \mathbb{R} = cl_c(\mathbb{Q}).$$

A TVS (X, τ) is called metrizable if there exists a metric $d : X \times X \rightarrow \mathbb{R}$ such that d -open sets in X coincide with τ -open sets in X . Now, we are going to show that (X, τ_c) is not metrizable when X is infinite-dimensional. Lemma 3.4 helps us to prove it. This lemma shows that every algebraic basis of the real vector space X is far from the origin with respect to τ_c topology.

Lemma 3.4. *Let $\{x_i\}_{i \in I}$ be an algebraic basis of X . Then $0 \notin cl_c(\{x_i\}_{i \in I})$.*

Proof. Define

$$A := \left\{ \sum_{i \in F} \lambda_i x_i - \sum_{i \in F} \mu_i x_i : \sum_{i \in F} (\lambda_i + \mu_i) = 1, \lambda_i, \mu_i > 0; \forall i \in F, \right. \\ \left. \text{and } F \text{ is a finite subset of } I \right\}.$$

We claim that the following propositions hold true;

- a. A is convex,
- b. $0 \in cor(A)$,
- c. $\{x_i\}_{i \in I} \cap A = \emptyset$.

To prove (a), assume that $x, y \in A$ and $t \in (0, 1)$. Then there exist positive scalars $\lambda_i, \mu_i, \bar{\lambda}_i, \bar{\mu}_i$ and finite sets $F, S \subseteq I$ such that;

$$x = \sum_{i \in F} \lambda_i x_i - \sum_{i \in F} \mu_i x_i, \quad y = \sum_{i \in S} \bar{\lambda}_i x_i - \sum_{i \in S} \bar{\mu}_i x_i,$$

$$\text{and} \quad \sum_{i \in F} (\lambda_i + \mu_i) = \sum_{i \in S} (\bar{\lambda}_i + \bar{\mu}_i) = 1.$$

Hence,

$$tx + (1-t)y = \sum_{i \in F \cup S} \hat{\lambda}_i x_i - \sum_{i \in F \cup S} \hat{\mu}_i x_i,$$

where

$$\hat{\lambda}_i = \begin{cases} t\lambda_i + (1-t)\bar{\lambda}_i & i \in F \cap S \\ t\lambda_i & i \in F \setminus S \\ (1-t)\bar{\lambda}_i & i \in S \setminus F \end{cases} \quad \hat{\mu}_i = \begin{cases} t\mu_i + (1-t)\bar{\mu}_i & i \in F \cap S \\ t\mu_i & i \in F \setminus S \\ (1-t)\bar{\mu}_i & i \in S \setminus F \end{cases}$$

Furthermore,

$$\begin{aligned} \sum_{i \in F \cup S} \hat{\lambda}_i + \hat{\mu}_i &= \sum_{i \in F \cap S} [t(\lambda_i + \mu_i) + (1-t)(\bar{\lambda}_i + \bar{\mu}_i)] + \sum_{i \in F \setminus S} t(\lambda_i + \mu_i) \\ &\quad + \sum_{i \in S \setminus F} (1-t)(\bar{\lambda}_i + \bar{\mu}_i) = \sum_{i \in F} t(\lambda_i + \mu_i) + \sum_{i \in S} (1-t)(\bar{\lambda}_i + \bar{\mu}_i) \\ &= t + (1-t) = 1. \end{aligned}$$

Therefore $tx + (1-t)y \in A$, and hence A is a convex set.

To prove (b), first notice that $\pm \frac{1}{2}x_i \in A$ for each $i \in I$; then due to the convexity of A , from (a) we get $0 \in \text{cor}(A)$, because of Lemma 2.1.

We prove assertion (c) by indirect proof. If $\{x_i\}_{i \in I} \cap A \neq \emptyset$, then

$$x_j = \sum_{i \in F} \lambda_i x_i - \sum_{i \in F} \mu_i x_i$$

for some $j \in I$, and some finite set $F \subseteq I$. Also, λ_i and μ_i values are positive and $\sum_{i \in F} (\lambda_i + \mu_i) = 1$. Since $\{x_i\}_{i \in F} \cup \{x_j\}$ is a linear independent set, we have $j \in F$, and also $1 = \lambda_j - \mu_j$. Hence $\lambda_j > 1$ which is a contradiction. This completes the proof of assertion (c).

Now, we prove the lemma invoking (a) – (c). Since A is convex, by theorem 3.2 and claim (b), $\text{cor}(A)$ is a τ_c -open neighborhood of 0. On the other hand, (c) implies $\text{cor}(A) \cap \{x_i\}_{i \in I} = \emptyset$. Thus, $0 \notin \text{cl}_c(\{x_i\}_{i \in I})$. \square

Theorem 3.8. *If X is infinite-dimensional, then (X, τ_c) is not metrizable.*

Proof. Suppose that (X, τ_c) is an infinite-dimensional metrizable TVS with metric $d(., .)$. Then for every $n \in \mathbb{N}$ choose $x_k \in X$, $k = 1, 2, \dots, n$ such that $d(0, x_k) < \frac{1}{k}$ and $\{x_1, x_2, \dots, x_n\}$ is linear independent. This process generates a linear independent sequence $\{x_n\}_{n \in \mathbb{N}}$ such that, $x_n \xrightarrow{\tau_c} 0$ (i.e. $\{x_n\}$ is τ_c -convergent to zero). Furthermore, this sequence can be extended to a basis of X , say $\{x_i\}_{i \in I}$. This makes a contradiction, according to Lemma 3.4, because $0 \in \text{cl}_c(\{x_i\}_{i \in I})$. \square

In every topological vector space, compact sets are closed and bounded, while the converse is not necessarily true. A topological vector space in which every closed and bounded set is compact, is called a *Hine-Borel* space. Although it is almost rare that an infinite-dimensional locally convex space be a Hine-Borel space (for example, normed spaces), the following result proves that (X, τ_c) is a Hine-Borel space.

Theorem 3.9. *(X, τ_c) enjoys the Hine-Borel property. Moreover, every τ_c -compact set in X lies in some finite-dimensional subspace of X .*

Proof. Let A be a τ_c -closed and τ_c -bounded subset of X . First we claim that the linear space $\langle A \rangle$, i.e. the smallest linear subspace containing A , is finite-dimensional. To see this, by indirect proof, assume that $\langle A \rangle$ is an infinite-dimensional subspace of X ; then there exists a linear independent sequence $\{x_n\}_{n=1}^\infty$ in A . Thus, the sequence $\{\frac{x_n}{n}\}_{n=1}^\infty$ is a linear independent set in X , and also $\{x_n\}_{n=1}^\infty$ is τ_c -bounded. Hence, $\frac{x_n}{n} \xrightarrow{\tau_c} 0$. Furthermore, X has a basis $\{x_i\}_{i \in I}$, containing $\{\frac{x_n}{n}\}_{n=1}^\infty$. This makes a contradiction, according to Lemma 3.4, because $0 \in \text{cl}_c(\{x_i\}_{i \in I})$. Hence, A is a closed and bounded set contained in a finite-dimensional space $\langle A \rangle$. Therefore, by traditional Hine-Borel theorem in finite-dimensional spaces, A is a compact set in $\langle A \rangle$, and hence A is a τ_c -compact set in X . \square

One of the important methods to realize the behavior of a topology defined on a nonempty set is to perceive (characterize) the convergent sequences. Assume that a sequence $\{x_n\}_{n=1}^\infty$ is τ_c -convergent to some $x \in X$ (i.e. $x_n \xrightarrow{\tau_c} x$). Thus, $\{x_n\}_{n=1}^\infty \cup \{x\}$ is a τ_c -compact set. Therefore, by Theorem 3.9, $\{x_n\}_{n=1}^\infty \cup \{x\}$ lies in a finite-dimensional subspace of X . This shows that, convergent sequences of (X, τ_c) are exactly norm-convergent sequences contained in finite-dimensional subspaces of X . Hence, every sequence with infinite-dimensional *span* could not be convergent. So, the convergence in (X, τ_c) is almost strict; however, this is not surprising because *strongest-topology* (τ_c) contains more number of open sets than any other topology which makes X into a locally convex space.

4 Applications

In this section, we address some important results in convex analysis and optimization under topological vector spaces which can be directly extended to real vector spaces, utilizing the main results presented in the previous section. Some of these results can be found in the literature with different complicated proofs. Subsection 4.1 is devoted to some Hahn-Banach type separation theorems.

4.1 Separation

Theorem 4.1. *Assume that A, B are two disjoint convex subsets of X , and $\text{cor}(A) \neq \emptyset$. Then there exist some $f \in X' \setminus \{0\}$ and some scalar $\alpha \in \mathbb{R}$ such that*

$$f(a) \leq \alpha \leq f(b), \quad \forall a \in A, b \in B. \quad (2)$$

Furthermore,

$$f(a) < \alpha, \quad \forall a \in \text{cor}(A). \quad (3)$$

Proof. Since (X, τ_c) is a TVS and $\text{int}_c(A) = \text{cor}(A) \neq \emptyset$, by a standard separation theorem on topological vector spaces (see [26]), there exists some $f \in (X, \tau_c)^*$ and some scalar $\alpha \in \mathbb{R}$ satisfying (2) and (3). This completes the proof according to Theorem 3.5. \square

Theorem 4.2. *Two disjoint convex sets $A, B \subseteq X$ are strongly separated by some $f \in X'$ if and only if there exists a convex absorbing set V in X such that $(A + V) \cap B = \emptyset$.*

Proof. It is easy to check that V and then $A + V$ are convex solid sets. Now it is enough to apply Theorem 4.1 for two sets $A + V$ and B . \square

In the following theorem, we brought some new conditions on two convex sets that guarantee their strongly separation by some $f \in X'$.

Theorem 4.3. *Suppose that $A, B \subseteq X$ are two disjoint convex sets such that, A is vectorial closed and relatively solid. If furthermore, $B \subseteq Y$, where Y is a finite-dimensional subspace of X , and B is compact in Y (note that Y is homeomorphic to $\mathbb{R}^{\dim Y}$), then A and B can be strongly separated by some $f \in X'$.*

Proof. Since A is convex and relatively solid then, by Theorem 3.7 we have $A = \text{vcl}(A) = \text{cl}_c(A)$. Thus A is τ_c -closed. Let $\{V_i\}_{i \in I}$ be a τ_c -open cover of B in X . Then $\{V_i \cap Y\}_{i \in I}$ is a norm-open cover of B in Y . Since B is compact in Y , $\{V_i \cap Y\}_{i \in I}$ and then $\{V_i\}_{i \in I}$ admit finite subcovers of B . Hence, B is τ_c -compact in X . Now for completing the proof, it is enough to apply the classic strong separation theorem for closed convex set A and convex compact set B in locally convex space (X, τ_c) ; see [26]. \square

Theorem 4.4. *(Separation of cones). Let M, K be two solid convex cones in X . If $M \cap \text{cor}(K) = \emptyset$, then there exists a functional $f \in X' \setminus \{0\}$ such that,*

$$f(m) \leq 0 \leq f(k) \quad \forall (k \in K, m \in M),$$

and furthermore,

$$f(k) > 0 \quad \forall k \in \text{cor}(K),$$

and

$$f(m) < 0 \quad \forall m \in \text{cor}(M).$$

Proof. This theorem results from Theorems 3.1, 3.2, and 3.5 in the present paper and standard separation theorem in TVSs (see [26]). \square

The following result is a Sandwich Theorem between two functions in real vector spaces without topology.

Theorem 4.5. *(Sandwich Theorem) Let X be a real vector space and let $f, g : X \rightarrow \mathbb{R}$ be convex and concave functions, respectively, such that $g(x) \leq f(x)$ for each $x \in X$. Then there exist some $l \in X'$ and $\alpha \in \mathbb{R}$ such that*

$$g(x) \leq l(x) + \alpha \leq f(x), \quad \forall x \in X.$$

Proof. Both functions f, g are τ_c -continuous by Lemma 2.3. Furthermore, these functions satisfy all assumptions of the Sandwich Theorem under TVS (X, τ_c) (see [27, Page 14]). This completes the proof. \square

Proposition 4.1. *Let A be a convex and relatively solid set in X . Then*

1. $\text{cor}(A^c) = \text{int}_c(A^c)$,
2. $\delta A = \text{vcl}(A) \setminus \text{cor}(A)$, where δA stands for the algebraic boundary of A .

Proof. 1. It is enough to prove $\text{cor}(A^c) \subseteq \text{int}_c(A^c)$. To see this, suppose $x \in \text{cor}(A^c)$. We claim that $x \notin \text{vcl}(A)$; otherwise there exist $d \in X$ and $\bar{\lambda} > 0$ such that $x + \lambda d \in A$ for each $\lambda \in [0, \bar{\lambda}]$ which contradicts $x \in \text{cor}(A^c)$. Therefore, $x \notin \text{vcl}(A)$. By Theorem 4.3 in the present paper, there exist $f \in X' \setminus \{0\}$ and $\alpha > 0$ such that

$$f(a) < \alpha < f(x) \quad \forall a \in A.$$

Therefore $U := \{z \in X : \alpha < f(z)\}$ is a convex and τ_c -open neighborhood of x satisfying $U \subseteq A^c$. So $x \in \text{int}_c(A^c)$.

2. According to definition of algebraic boundary of A , we have $\delta A = (\text{cor}(A^c) \cup \text{cor}(A))^c$. Using Part 1, we obtain

$$\delta A = (\text{int}_c(A^c) \cup \text{int}_c(A))^c = [\text{int}_c(A^c)]^c \cap [\text{int}_c(A)]^c = \text{cl}_c(A) \setminus \text{int}_c(A).$$

Now, using Theorems 3.2 and 3.7, we conclude $\delta A = \text{vcl}(A) \setminus \text{cor}(A)$. \square

We close this subsection by a nonconvex separation theorem in real vector spaces. This theorem is a direct consequence of a nonconvex separation theorem presented in [13].

Definition 4.1. Let D be a nonempty subset of X . A real-valued function $g : X \rightarrow \mathbb{R}$ is called

- 1) D -monotone if $x_2 \in x_1 + D$ implies $g(x_1) \leq g(x_2)$.
- 2) strictly D -monotone, if $x_2 \in x_1 + D \setminus \{0\}$ implies $g(x_1) < g(x_2)$.

Theorem 4.6. *Suppose that the following conditions are satisfied:*

- \square C is a proper convex and algebraic open subset of X ;
- \square There exists $k \in X$ such that $\text{vcl}(C) - \alpha k \subseteq \text{vcl}(C)$ for each $\alpha > 0$;
- \square $X = \bigcup \{\text{vcl}(C) + \alpha k : \alpha \in \mathbb{R}\}$;
- \square $A \subset X$ is a nonempty subset of X .

Then we have:

- (a) $A \cap C = \emptyset$ if and only if there exists a convex (τ_c -continuous) and onto function $g : X \rightarrow \mathbb{R}$ such that

$$g(A) \geq 0, \quad g(\text{int}_c A) > 0, \quad g(C) < 0, \quad g(\delta C) = 0,$$

where δC stands for algebraic boundary of C , i.e. $\delta C = \text{vcl}(C) \setminus \text{cor}(C)$.

- (b) If $B \subset X$ and $\delta C - B \subset C$, then the function g in part (a) can be chosen such that it is also B -monotone.
- (c) If $B \subset X$ and $\delta C - (B \setminus \{0\}) \subset C$, then one can construct function g in part (a) such that it is also strictly B -monotone.
- (d) If $\delta C + \delta C \subset \text{vcl}(C)$, then g in part (a) can be chosen such that it is subadditive.

Proof. The desired results are obtained by [13, Theorem 2.2], because (X, τ_c) is a TVS, $\text{int}_c(C) = \text{cor}(C)$, and $\text{cl}_c(C) = \text{vcl}(C)$. \square

4.2 Vector optimization

In this subsection, we address some results in vector optimization under real vector spaces, to show that various important results existing in the literature can be extended from TVSs to real vector spaces utilizing the main results of the present paper.

Let Z, X be two real vector spaces such that X is partially ordered by a nontrivial ordering convex cone K . Notice that $0 \notin \text{cor}(K)$ because $K \neq Y$ (K is nontrivial). Consider the following vector optimization problem:

$$(\text{VOP}) \quad K - \text{Min} \{f(x) : x \in \Omega\},$$

where $\Omega \subseteq Z$ is a nonempty set. Here, Ω is the feasible set and $f : Z \rightarrow X$ is the objective function.

We say that a nonempty set $B \subseteq X$ is a base of the cone $K \subseteq X$ if $0 \notin B$ and for each $k \in K \setminus \{0\}$ there are unique $b \in B$ and unique $t > 0$ such that $k = tb$.

Throughout this section, we assume that K is a nontrivial convex ordering cone with a convex base B . It is not difficult to show that convex ordering cone K is pointed if it has a convex base. Also, it can be shown that $0 \notin \text{vcl}(B)$. Hence, the considered ordering cone K is nontrivial, convex and pointed.

Recall that a set $A \subseteq X$ is called algebraic open if $\text{cor}(A) = A$. Also, it is vectorially closed if $\text{vcl}(A) = A$.

Definition 4.2. A feasible solution $x_0 \in \Omega$ is called an efficient (EFF) solution of (VOP) with respect to K if $(f(\Omega) - f(x_0)) \cap (-K) = \{0\}$.

The following definition extends weak efficiency notion from topological vector spaces to vector spaces; see [1].

Definition 4.3. Assuming $\text{core}(K) \neq \emptyset$, the feasible solution $x_0 \in \Omega$ is called a vectorial weakly efficient (VWEFF) solution of (VOP) with respect to K if $(f(\Omega) - f(x_0)) \cap (-\text{cor}(K)) = \emptyset$.

One of the solution concepts which plays an important role in vector optimization, from both theoretical and practical points of view, is the proper efficiency notion [9]. This concept has been introduced to eliminate the efficient solutions with unbounded trade offs [12]. There are different definitions for proper efficiency in the literature; see e.g. [5, 6, 9, 12, 14, 15, 17, 18, 21, 28] and the references therein. Definition 4.4 extends the concept of proper efficiency given by Guerraggio and Luc [14]; see [14, Definition 2.1].

Definition 4.4. $x_0 \in \Omega$ is called a vectorial proper (VP) solution of (VOP) if there exists a convex algebraic open set V containing zero such that $\text{cone}(B + V) \neq X$ and x_0 is efficient with respect to $\text{cone}(B + V)$.

In the following, we extend proper efficiency notion in the Henig's sense under vector spaces.

Definition 4.5. $x_0 \in \Omega$ is called a vectorial Henig (VH) solution of (VOP) if there exists an ordering pointed convex cone C such that $K \setminus \{0\} \subseteq \text{cor}(C)$ and x_0 is efficient with respect to C .

Two of the most important definitions of proper efficiency have been given by Hurwicz [17] and Benson [5]. The following definition extends these notions under vector spaces.

Definition 4.6. [3] $x_0 \in \Omega$ is called a Hurwicz vectorial (HuV) proper efficient solution of (VOP) if

$$vcl\left(\text{conv}\left(\text{cone}\left((f(\Omega) - f(x_0)) \cup K\right)\right)\right) \cap (-K) = \{0\};$$

and $x_0 \in \Omega$ is called a Benson vectorial (BeV) proper efficient solution of (VOP) if

$$vcl\left(\text{cone}\left(f(\Omega) - f(x_0) + K\right)\right) \cap (-K) = \{0\}.$$

The following result gives a connection between the above-defined notions. Since (X, τ_c) is a TVS, the following theorem results from corresponding results in topological vectors spaces (see e.g. [9, 15]) and the results of the present paper.

Theorem 4.7. (i) Each BeV proper efficient solution is an efficient solution.

(ii) Each HuV proper efficient solution is a BeV proper efficient solution. The converse holds under the convexity of $f(\Omega) + K$.

(iii) Each VP solution is a VH solution.

(iv) Each VH efficient solution is a BeV proper efficient solution.

Proof. Part (i) is clear. For each $x_0 \in \Omega$, the inclusion

$$\text{cone}\left(f(\Omega) - f(x_0)\right) \subseteq \text{conv}\left(\text{cone}\left((f(\Omega) - f(x_0)) \cup K\right)\right)$$

always holds and two sets are equal if $f(\Omega) + K$ is convex. This proves part (ii).

To prove part (iii), let x_0 be a VP solution. Then there exists a convex set V containing zero such that $\text{cor}(V) = V$ and $\text{cone}(B + V) \neq X$. Furthermore, x_0 is efficient with respect to $\text{cone}(B + V)$. Setting $C := \text{cone}(B + V)$, the set C is a convex cone. Considering nonzero $k \in K$, there exist scalar $\lambda > 0$ and vector $b \in B$ such that $k = \lambda b$. Since V is convex and $0 \in V = \text{cor}(V)$, for each $y \in X$ there exists $t' > 0$ such that $\frac{t'}{\lambda}y \in V$ for each $t \in [0, t']$. Hence,

$$k = \lambda b = \lambda(b + 0) \in \text{cone}(B + V) = C$$

and

$$k + ty = \lambda\left(b + \frac{t}{\lambda}y\right) \in \text{cone}(B + V) = C, \quad \forall t \in [0, t'].$$

These imply $k \in \text{cor}(C)$. Therefore, $K \setminus \{0\} \subseteq \text{cor}(C)$ and x_0 is efficient with respect to C . Hence, x_0 is a VH solution.

Part (iv) results from [9, Proposition 2.4.11], Theorem 3.2, and $vcl(\cdot) \subseteq cl_c(\cdot)$. \square

Now, we use some scalarization problems. Given the convex base B of K , we define

$$K^{ss} := \{l \in X' : \inf_{b \in B} \langle l, b \rangle > 0\}.$$

This set has been studied in some papers including [14]. The following lemma provides a sufficient condition for nonemptiness of K^{ss} .

Lemma 4.1. *$K^{ss} \neq \emptyset$ when K has a convex and relatively solid base.*

Proof. Let B be a convex and relatively solid base of K . Then $0 \notin \text{vcl}(B)$, and thus by Theorem 4.3, there is a nonzero linear functional $l \in X'$ and $\alpha > 0$ such that $\langle l, 0 \rangle < \alpha < \langle l, x \rangle$ for every $x \in \text{vcl}(B)$. Therefore, $\inf_{b \in B} \langle l, b \rangle \geq \alpha > 0$, which means $l \in K^{ss}$. \square

Now considering a linear functional l belong to K^+ , K^{+s} , or K^{ss} , we define the following auxiliary scalarization problem:

$$\min_{x \in \Omega} \langle l, f(x) \rangle. \quad (4)$$

The set of optimal solutions of Problem (4) is denoted by O_l . In fact, Problem (4) comes from the weighted sum scalarization method which is a popular technique in multiobjective optimization [18]. Now, we define some optimal sets as follows (see also [9, 14]).

$$O^+ := \{x_0 \in \Omega : \exists l \in K^+ \text{ s.t. } x_0 \in O_l\},$$

$$O^s := \{x_0 \in \Omega : \exists l \in K^{+s} \text{ s.t. } x_0 \in O_l\},$$

$$O^{ss} := \{x_0 \in \Omega : \exists l \in K^{ss} \text{ s.t. } x_0 \in O_l\}.$$

It is clear that $O^{ss} \subseteq O^s \subseteq O^+$ and the members of O^s are efficient with respect to K . The following result provides a sufficient condition for VH solutions utilizing O^s .

Theorem 4.8. *If $x_0 \in O^s$, then x_0 is a VH solution.*

Proof. Since (X, τ_c) is a TVS, this theorem results from [9, Proposition 2.4.15] and Theorems 3.2 and 3.5 in the present paper. \square

Now, we state a theorem which has been proved by Adan and Novo [2]. This theorem will be useful for getting an important corollary.

Theorem 4.9. *Let K be solid and vectorially closed. Let $x_0 \in \Omega$ and $\text{vcl}\left(\text{cone}(f(\Omega) - f(x_0)) + K\right)$ be convex. If x_0 is a BeV solution to (VOP), then $x_0 \in O^s$.*

Theorems 4.8 and 4.9 lead to the following corollary which completes the relationship between BeV solutions and VH ones.

Corollary 4.1. *Suppose that K is solid and vectorially closed. Furthermore, assume that $x_0 \in \Omega$ and $\text{vcl}\left(\text{cone}(f(\Omega) - f(x_0)) + K\right)$ is convex. If x_0 is a BeV solution, then x_0 is a VH solution.*

Proposition 4.2. (i) *If $x_0 \in O^{ss}$, then x_0 is a VP solution.*

(ii) *If $x_0 \in \Omega$ is a VP solution, and $f(\Omega) + K$ is convex and relatively solid, then $x_0 \in O^{ss}$.*

Proof. This proposition results from [14, Proposition 2.1] and Theorems 3.2, 3.5, and 3.7 in the present paper. \square

Below a diagram is given presenting the proved relationships between (weak) efficient solutions and proper efficient points in different senses. In this diagram, “s.a.” means under “some assumptions” which can be seen from the corresponding theorem.

$$\begin{array}{ccccccc} & & O^s & & & & \\ & & \Downarrow & & & & \\ O^{ss} & \xrightleftharpoons{\text{s.a.}} & VP & \Longrightarrow & VH & \xrightleftharpoons{\text{s.a.}} & BeV \Longrightarrow EFF \Longrightarrow WEFF \\ & & & & \Uparrow \Downarrow \text{s.a.} & & \\ & & & & HuV & & \end{array}$$

Now, we continue with more efficiency notions. These notions have been investigated by Borwein and Zhuang [7] and Guerraggio and Luc [14] under normed and topological vector spaces. We start with extending these definitions from locally convex TVSs to real vector spaces.

Definition 4.7. $x_0 \in \Omega$ is called

(i) a supper efficient solution, if for each convex algebraic open set V containing zero, there exists a convex algebraic open set U containing zero such that

$$vcl\left(\text{cone}\left(f(x_0) - f(\Omega)\right)\right) \cap (K + U) \subseteq V.$$

(ii) a strictly efficient solution, if there exists a convex algebraic open set V containing zero such that

$$vcl\left(\text{cone}\left(f(x_0) - f(\Omega)\right)\right) \cap (B + V) = \emptyset.$$

(iii) a strongly efficient solution, if for each algebraic continuous $x^* \in X'$ there are convex algebraic open sets U, V containing zero, such that $\langle x^*, \cdot \rangle$ is bounded on

$$\text{cone}\left(f(x_0) - f(\Omega)\right) \cap (U + \text{cone}(B + V)).$$

The following result gives some connections between these different efficiency concepts and also with the concepts studied so far.

Theorem 4.10. Let B be τ_c -bounded, $x_0 \in \Omega$ and $\text{cone}\left(f(x_0) - f(\Omega)\right)$ be relatively solid and convex.

The following assertions are equivalent:

- (i) x_0 is a supper efficient solution;
- (ii) x_0 is a strictly efficient solution;
- (iii) x_0 is a strongly efficient solution;
- (iv) x_0 is a VP solution.

Proof. This theorem results from [14, Proposition 2.2] and Theorems 3.2, 3.5, and 3.7 in the present paper. \square

As a result of Theorem 4.6, the following theorem gives a characterization of weakly efficient solutions in nonconvex vector optimization by means of convex and (or) sublinear functions. The following theorem is, in fact, a direct generalization of [13, Corollary 3.1] to vector spaces without topology.

Theorem 4.11. Let K be convex and solid and $x_0 \in \Omega$. Then

1. x_0 is a weakly efficient solution of (VOP) if and only if there exists a convex onto function $g : X \rightarrow \mathbb{R}$ which is strictly $\text{cor}(K)$ -monotone and

$$g(f(x_0)) = 0, \quad g(f(\Omega)) \geq 0, \quad g(\text{int}_c f(\Omega)) > 0,$$

$$g(f(x_0) - K) \leq 0, \quad g(f(x_0) - \text{cor}(K)) < 0, \quad g(f(x_0) - \delta K) = 0.$$

If $f(x_0) = 0$, then g can be chosen such that it is subadditive.

2. x_0 is a weakly efficient solution of (VOP) if and only if there exists a sublinear onto function $g : X \rightarrow \mathbb{R}$ which is strictly $\text{cor}(K)$ -monotone and

$$g(f(\Omega) - f(x_0)) \geq 0, \quad g(\text{int}_c f(\Omega) - f(x_0)) > 0, \quad g(-K) < 0,$$

$$g(-\delta K) = 0, \quad g(-\text{cor}(K)) < 0, \quad g(K) \geq 0, \quad g(\text{cor}K) > 0.$$

Proof. The desired results are obtained by [13, Corollary 3.1], because (X, τ_c) is a TVS, $\text{int}_c(K) = \text{cor}(K)$, and $\text{cl}_c(K) = vcl(K)$. \square

Remark 4.1. The results of this section are some selected important issues which can be easily extended from TVSs to real vector spaces by the main results of the present paper. Such extension can be done for many other results in Optimization and Convex Analysis.

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